Integrated Classical Mechanics

Complete Solutions

Coverage: Newtonian flow, variational mechanics, generalized coordinates, Noether charges, Hamiltonian flow, Liouville's theorem, Poisson brackets, generators, symmetry groups, and angular momentum.

目录

1. Newtonian Motion as a Flow **1**

2. From Stationary Action to a Boundary-Value Solution **2**

3. Generalized Coordinates: Bead on a Rotating Hoop **2**

4. Constraints and Multipliers: A Sliding Wedge **3**

5. Noether's Theorem in Calculational Form **3**

6. Central Forces: Reduction Before Solution **4**

7. One System in Three Languages: The Double Pendulum **4**

8. When Canonical Momentum Is Not $m\dot{q}$ **4**

9. Hamiltonian Flow of the Harmonic Oscillator **5**

10. Pendulum Phase Portrait and Separatrix **5**

11. Liouville's Theorem and Its Failure for Damping **6**

12. Poisson Brackets from Hamilton's Equations **6**

13. Generators of Translations and Rotations **6**

14. $SO(3)$, Its Lie Algebra, and Finite Rotations **7**

15. Hidden Symmetry of the Kepler Problem **7**

16. Discrete Symmetry and Normal Modes **8**

17. Canonical Transformations from a Generator **8**

18. Rigid-Body Dynamics from the Angular-Momentum Algebra **8**

1. Newtonian Motion as a Flow

(a) Newton's equation is $m\ddot{x} = -m\omega^2 x - \lambda x^3$. Thus

$$\dot{\boldsymbol{z}} = \boldsymbol{F}(\boldsymbol{z}), \qquad \begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x - (\lambda/m)x^3 \end{pmatrix}.$$

The only fixed point is $(0,0)$. Its Jacobian is $D\boldsymbol{F}(0,0) = \left(\begin{smallmatrix} 0 & 1 \\ -\omega^2 & 0 \end{smallmatrix}\right)$, with eigenvalues $\pm i\omega$; the linearized fixed point is a center.

(b) Along the flow,

$$\dot{E} = mv\dot{v} + (m\omega^2 x + \lambda x^3)\dot{x} = v[m\dot{v} + m\omega^2 x + \lambda x^3] = 0.$$

At a turning point $x = A$, $E = V(A)$. On one quarter-cycle,

$$\dot{x} = \sqrt{\frac{2}{m}[V(A) - V(x)]}, \qquad T(A) = 4\sqrt{\frac{m}{2}}\int_0^A \frac{dx}{\sqrt{V(A) - V(x)}}.$$

(c) Put $x = A\sin u$. Since

$$V(A) - V(x) = A^2\cos^2 u\left[\frac{1}{2}m\omega^2 + \frac{\lambda A^2}{4}(1 + \sin^2 u)\right],$$

we get, with $\delta = \lambda A^2/(2m\omega^2)$,

$$T = \frac{4}{\omega}\int_0^{\pi/2}[1 + \delta(1 + \sin^2 u)]^{-1/2}du.$$

Using $(1 + y)^{-1/2} = 1 - y/2 + O(y^2)$ and $\int_0^{\pi/2}\sin^2 u\,du = \pi/4$ gives

$$T = \frac{2\pi}{\omega}\left[1 - \frac{3\lambda A^2}{8m\omega^2} + O\left(\frac{\lambda^2 A^4}{m^2\omega^4}\right)\right].$$

2. From Stationary Action to a Boundary-Value Solution

(a) Differentiating under the integral,

$$\delta S = \int_{t_1}^{t_2}(L_q\eta + L_{\dot{q}}\dot{\eta})\,dt = [L_{\dot{q}}\eta]_{t_1}^{t_2} + \int_{t_1}^{t_2}\left(L_q - \frac{d}{dt}L_{\dot{q}}\right)\eta\,dt.$$

The endpoint condition kills the boundary term. Since the interior η is arbitrary, $dL_{\dot{q}}/dt - L_q = 0$.

(b) The equation is $m\ddot{y} = -mg$. Hence $y = -gt^2/2 + at + b$. The endpoint conditions give $b = 0$ and $a = h/T + gT/2$, so

$$y_\star(t) = \left(\frac{h}{T} + \frac{gT}{2}\right)t - \frac{g}{2}t^2.$$

(c) Substitute $y = y_\star + \eta$:

$$S[y] - S[y_\star] = \int_0^T\left(m\dot{y}_\star\dot{\eta} - mg\eta + \frac{m}{2}\dot{\eta}^2\right)dt.$$

The first two terms equal $[m\dot{y}_\star\eta]_0^T - \int_0^T(m\ddot{y}_\star + mg)\eta\,dt = 0$. Thus the stated identity holds. It is nonnegative and vanishes only for $\eta = 0$; the stationary path is therefore the unique global minimum in this admissible class.

3. Generalized Coordinates: Bead on a Rotating Hoop

(a) Direct differentiation gives $\dot{\boldsymbol{r}}^2 = R^2(\dot{\theta}^2 + \Omega^2\sin^2\theta)$ and $V = mgz = -mgR\cos\theta$. Therefore

$$L = \frac{1}{2}mR^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + mgR\cos\theta.$$

(b) The equation and effective potential are

$$\ddot{\theta} = \sin\theta(\Omega^2\cos\theta - g/R), \qquad V_{\text{eff}} = -\frac{1}{2}mR^2\Omega^2\sin^2\theta - mgR\cos\theta.$$

Equilibria are $\theta = 0, \pi$. If $\Omega^2 \geq g/R$, there are also two equilibria $\cos\theta_\star = g/(R\Omega^2)$. They are born at $\Omega_c = \sqrt{g/R}$.

(c) Since stable equilibria are minima of V_{eff},

$$V''_{\text{eff}}(0) = mR^2(g/R - \Omega^2), \quad V''_{\text{eff}}(\pi) = -mR^2(g/R + \Omega^2).$$

Thus 0 is stable for $\Omega < \Omega_c$ with $\omega_{0,\text{small}} = \sqrt{g/R - \Omega^2}$, becomes unstable above threshold, and π is always unstable. At the two off-axis equilibria, $V''_{\text{eff}} = mR^2\Omega^2\sin^2\theta_\star > 0$, so

$$\omega_\star = \Omega\sin\theta_\star = \sqrt{\Omega^2 - \frac{g^2}{R^2\Omega^2}}.$$

At threshold the quadratic term vanishes and linear stability is inconclusive.

4. Constraints and Multipliers: A Sliding Wedge

(a) Here $f_x = \tan\alpha$, $f_y = 1$, $f_X = -\tan\alpha$. Hence

$$m\ddot{x} = \Lambda\tan\alpha, \qquad m\ddot{y} + mg = \Lambda, \qquad M\ddot{X} = -\Lambda\tan\alpha, \qquad f = 0.$$

(b) With $x = X + s\cos\alpha$, $y = y_0 - s\sin\alpha$,

$$L = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{s}\cos\alpha + \frac{1}{2}m\dot{s}^2 + mgs\sin\alpha + \text{const.}$$

The two equations are

$$(M+m)\ddot{X} + m\ddot{s}\cos\alpha = 0, \qquad \ddot{s} + \ddot{X}\cos\alpha = g\sin\alpha.$$

Therefore

$$\boxed{\ddot{s} = \frac{(M+m)g\sin\alpha}{M + m\sin^2\alpha}}, \qquad \boxed{\ddot{X} = -\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}}.$$

(c) The block's constraint force is $\Lambda(\tan\alpha, 1) = N(\sin\alpha, \cos\alpha)$, so $N = \Lambda/\cos\alpha$. Using $m\ddot{y} + mg = \Lambda$ and $\ddot{y} = -\ddot{s}\sin\alpha$,

$$N = \frac{mMg\cos\alpha}{M + m\sin^2\alpha}.$$

The conserved horizontal momentum is $P_X = (M+m)\dot{X} + m\dot{s}\cos\alpha$; its derivative is the first reduced equation. Because the reduced L has no explicit time dependence,

$$E = \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{s}\cos\alpha + \frac{1}{2}m\dot{s}^2 - mgs\sin\alpha + \text{const}$$

is conserved; direct differentiation reduces to the two equations above.

5. Noether's Theorem in Calculational Form

(a) Along a solution,

$$\delta L = \sum_i \left(L_{q_i} X_i + p_i \frac{dX_i}{dt} \right) = \sum_i \left(\dot{p}_i X_i + p_i \dot{X}_i \right) = \frac{d}{dt}\sum_i p_i X_i.$$

Equating this to dF/dt gives $\dot{Q} = 0$.

(b) For translations, $X_a = \boldsymbol{a}$, $F = 0$, hence $Q = \boldsymbol{a}\cdot\sum_a \boldsymbol{p}_a$ and $\boldsymbol{P} = \sum_a \boldsymbol{p}_a$ is conserved. For rotations, $X_a = \boldsymbol{n}\times\boldsymbol{r}_a$, so

$$Q = \sum_a \boldsymbol{p}_a \cdot (\boldsymbol{n}\times\boldsymbol{r}_a) = \boldsymbol{n}\cdot\sum_a (\boldsymbol{r}_a\times\boldsymbol{p}_a).$$

Thus $\boldsymbol{L} = \sum_a \boldsymbol{r}_a\times\boldsymbol{p}_a$ is conserved.

(c) Under $\delta\boldsymbol{r}_a = \varepsilon\boldsymbol{u}t$,

$$\delta L = \varepsilon\sum_a m_a\dot{\boldsymbol{r}}_a\cdot\boldsymbol{u} = \varepsilon\frac{d}{dt}\left(\boldsymbol{u}\cdot\sum_a m_a\boldsymbol{r}_a\right).$$

Thus $F = \boldsymbol{u}\cdot\sum_a m_a\boldsymbol{r}_a$, and arbitrary $\boldsymbol{u}$ yields

$$\boldsymbol{K} = t\boldsymbol{P} - \sum_a m_a\boldsymbol{r}_a = \text{constant}.$$

Equivalently, $\sum m_a\boldsymbol{r}_a - t\boldsymbol{P}$ is conserved with the opposite sign.

6. Central Forces: Reduction Before Solution

(a)

$$L = \frac{\mu}{2}(\dot r^2 + r^2\dot\phi^2) + \frac{k}{r}, \qquad p_\phi = \mu r^2\dot\phi = \ell.$$

Time-translation invariance gives

$$E = \frac{\mu}{2}\dot r^2 + \frac{\ell^2}{2\mu r^2} - \frac{k}{r}.$$

(b) $V'_{\text{eff}} = -\ell^2/(\mu r^3) + k/r^2 = 0$, so

$$r_c = \frac{\ell^2}{\mu k}, \qquad E_c = -\frac{\mu k^2}{2\ell^2}.$$

Since $V''_{\text{eff}}(r_c) = k/r_c^3$, the radial frequency is $\omega_r = \sqrt{k/(\mu r_c^3)}$. The orbital frequency is $\omega_\phi = \ell/(\mu r_c^2)$; substitution of r_c shows $\omega_r = \omega_\phi$.

(c) Using $\dot r = (dr/d\phi)\ell/(\mu r^2) = -(\ell/\mu)u'$, the radial equation gives

$$u'' + u = \frac{\mu k}{\ell^2}.$$

Hence $u = p^{-1}[1 + e\cos(\phi - \phi_0)]$, where

$$p = \frac{\ell^2}{\mu k}, \qquad e = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}.$$

$E < 0$ gives an ellipse ($e < 1$, including the circle), $E = 0$ a parabola, and $E > 0$ a hyperbola.

7. One System in Three Languages: The Double Pendulum

(a) Positions are

$$\boldsymbol{r}_1 = \ell(\sin\theta_1, -\cos\theta_1), \quad \boldsymbol{r}_2 = \boldsymbol{r}_1 + \ell(\sin\theta_2, -\cos\theta_2).$$

Thus $v_1^2 = \ell^2\dot\theta_1^2$ and $v_2^2 = \ell^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2\cos(\theta_1 - \theta_2))$. Together with $V = -mg\ell(2\cos\theta_1 + \cos\theta_2)$, this gives the stated L.

(b) To quadratic order,

$$L_2 = \frac{1}{2}\dot{\boldsymbol\theta}^T\mathbf{M}\dot{\boldsymbol\theta} - \frac{1}{2}\boldsymbol\theta^T\mathbf{K}\boldsymbol\theta + \text{const}, \quad \mathbf{M} = m\ell^2\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix}, \quad \mathbf{K} = mg\ell\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}.$$

$\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ gives, with $s = \omega^2\ell/g$, $s^2 - 4s + 2 = 0$. Therefore

$$\omega_\pm^2 = \frac{g}{\ell}(2 \pm \sqrt{2}), \qquad \boldsymbol{v}_- \propto (1, \sqrt{2}), \quad \boldsymbol{v}_+ \propto (1, -\sqrt{2}).$$

(c) $\boldsymbol{p} = \mathbf{M}\dot{\boldsymbol\theta}$ and

$$H_2 = \frac{1}{2}\boldsymbol{p}^T\mathbf{M}^{-1}\boldsymbol{p} + \frac{1}{2}\boldsymbol\theta^T\mathbf{K}\boldsymbol\theta, \qquad \mathbf{M}^{-1} = \frac{1}{m\ell^2}\begin{pmatrix}1 & -1\\ -1 & 2\end{pmatrix}.$$

Hamilton's equations are $\dot{\boldsymbol\theta} = \mathbf{M}^{-1}\boldsymbol{p}$, $\dot{\boldsymbol{p}} = -\mathbf{K}\boldsymbol\theta$. Differentiating the first and using the second gives $\mathbf{M}\ddot{\boldsymbol\theta} + \mathbf{K}\boldsymbol\theta = 0$.

8. When Canonical Momentum Is Not $m\dot q$

(a)

$$\frac{d}{dt}(\alpha\dot q^3) + kq = 0 \quad \Longrightarrow \quad 3\alpha\dot q^2\ddot q + kq = 0,$$

and

$$E = \dot q(\alpha\dot q^3) - L = \frac{3\alpha}{4}\dot q^4 + \frac{k}{2}q^2.$$

(b) $p = \alpha\dot{q}^3$, so $\dot{q} = \mathrm{cbrt}(p/\alpha)$ and

$$H = p\dot{q} - L = \frac{3}{4\alpha^{1/3}}|p|^{4/3} + \frac{k}{2}q^2.$$

Then

$$\dot{q} = H_p = \mathrm{cbrt}(p/\alpha), \qquad \dot{p} = -H_q = -kq.$$

Since $p = \alpha\dot{q}^3$, the second equation is precisely $d(\alpha\dot{q}^3)/dt = -kq$.

(c) At $q = A$, $E = kA^2/2$. Hence

$$\dot{q} = \left[\frac{2k}{3\alpha}(A^2 - q^2)\right]^{1/4},$$

which gives the stated formula after $q = Au$. Moreover,

$$\int_0^1 (1-u^2)^{-1/4}du = \frac{1}{2}B\left(\frac{1}{2},\frac{3}{4}\right) = \frac{\sqrt{\pi}\,\Gamma(3/4)}{2\Gamma(5/4)}.$$

Thus $T \propto A^{1/2}$ and $\omega_{\mathrm{orbit}} = 2\pi/T \propto A^{-1/2}$.

9. Hamiltonian Flow of the Harmonic Oscillator

(a) $\dot{q} = p/m$, $\dot{p} = -m\omega^2 q$, so

$$S(t) = \begin{pmatrix} \cos\omega t & (m\omega)^{-1}\sin\omega t \\ -m\omega\sin\omega t & \cos\omega t \end{pmatrix}.$$

(b) Direct multiplication gives $S^T J S = J$ and $\det S = 1$. The angle-addition formulas give $S(t+s) = S(t)S(s)$. Equivalently, $S(t) = e^{tA}$ with $A = \left(\begin{smallmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{smallmatrix}\right)$, so the group law is immediate.

(c) In (Q,P),

$$H = \frac{\omega}{2}(P^2+Q^2), \qquad \begin{pmatrix} Q(t) \\ P(t) \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{pmatrix}\begin{pmatrix} Q_0 \\ P_0 \end{pmatrix}.$$

The ellipse in (q,p) has semiaxes $\sqrt{2E/(m\omega^2)}$ and $\sqrt{2mE}$, hence area $2\pi E/\omega$. Because $\oint p\,dq$ equals the enclosed oriented area,

$$I = \frac{1}{2\pi}\oint p\,dq = \frac{E}{\omega}$$

(choosing the positive action orientation).

10. Pendulum Phase Portrait and Separatrix

(a) Fixed points require $p = 0$ and $\sin\theta = 0$. Thus $(2\pi n, 0)$ are centers: locally $\ddot{\eta} + (g/\ell)\eta = 0$. The points $((2n+1)\pi, 0)$ are saddles: $\ddot{\eta} - (g/\ell)\eta = 0$.

(b) At $E_s = 2mg\ell$,

$$p = \pm 2m\ell\sqrt{g\ell}\cos(\theta/2), \qquad \dot{\theta} = \pm 2\omega_0\cos(\theta/2)$$

on $-\pi < \theta < \pi$. Set $u = \tan(\theta/4 + \pi/4)$; separation gives $d(\ln u)/dt = \omega_0$, so $u = e^{\omega_0(t-t_0)}$ and

$$\theta = 4\arctan e^{\omega_0(t-t_0)} - \pi.$$

(c) Energy conservation with $E = mg\ell(1-\cos\theta_0)$ gives

$$\dot{\theta}^2 = 4\frac{g}{\ell}\left(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}\right).$$

In a quarter-period put $\sin(\theta/2) = k\sin\psi$, $k = \sin(\theta_0/2)$; this yields the stated T. As $k \to 1$, the integrand near $\psi = \pi/2$ behaves like $[(1-k^2) + (\pi/2 - \psi)^2]^{-1/2}$, whose integral grows logarithmically. Thus $K(k) \sim \log[4/\sqrt{1-k^2}]$ and $T \to \infty$.

11. Liouville's Theorem and Its Failure for Damping

(a) $\boldsymbol{v}_H = (H_{p_1}, \ldots, H_{p_n}, -H_{q_1}, \ldots, -H_{q_n})$, hence

$$\nabla \cdot \boldsymbol{v}_H = \sum_i (H_{q_i p_i} - H_{p_i q_i}) = 0.$$

Explicit t dependence does not affect this cancellation.

(b) The variational equation is $\dot{M} = (D\boldsymbol{v}_H)M$. Thus $M^{-1}\dot{M} = M^{-1}(D\boldsymbol{v}_H)M$ and its trace is $\operatorname{tr} D\boldsymbol{v}_H = \nabla \cdot \boldsymbol{v}_H = 0$. Jacobi's formula gives $d(\det M)/dt = 0$. Since $M(0) = I$, $\det M(t) = 1$.

(c) The damped vector field has

$$\nabla \cdot \boldsymbol{v} = \partial_q(p/m) + \partial_p(-m\omega^2 q - \gamma p) = -\gamma.$$

Therefore an infinitesimal area satisfies $\dot{A} = -\gamma A$ and $A(t) = A(0)e^{-\gamma t}$. A canonical Hamiltonian flow in (q, p) must have zero divergence, whereas this one has $-\gamma$; no time-independent canonical $H(q, p)$ can generate it (indeed, neither can an explicitly time-dependent canonical $H(q, p, t)$ in the same two variables).

12. Poisson Brackets from Hamilton's Equations

(a) The chain rule and Hamilton's equations give

$$\frac{df}{dt} = f_t + \sum_i (f_{q_i} H_{p_i} - f_{p_i} H_{q_i}) = f_t + \{f, H\}.$$

Bilinearity follows from linearity of differentiation; swapping f, g changes the sign; and the product rule gives $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

(b) With the stated J, $\{f, g\} = (\nabla f)^T J \nabla g$. In indices, $\{f, g\} = f_{,a} J_{ab} g_{,b}$. Expanding the cyclic Jacobi sum produces terms with second derivatives and terms with first derivatives of J. The latter vanish because J is constant. The remaining terms cancel pairwise because $J_{ab} = -J_{ba}$ while $f_{,ab} = f_{,ba}$ (and similarly for g, h). Hence the Jacobi identity holds.

(c) Jacobi gives

$$\frac{d}{dt}\{A, B\} = \{\{A, B\}, H\} = \{A, \{B, H\}\} - \{B, \{A, H\}\} = 0.$$

For the isotropic oscillator $H = \sum_i [p_i^2/(2m) + m\omega^2 q_i^2/2]$, the quantities $L_{ij} = q_i p_j - q_j p_i$ obey $\{L_{ij}, H\} = 0$ and close:

$$\{L_{ij}, L_{kl}\} = delta_{ik} L_{jl} - \delta_{il} L_{jk} - \delta_{jk} L_{il} + \delta_{jl} L_{ik},$$

the $\mathfrak{so}(n)$ algebra (up to the chosen index convention).

13. Generators of Translations and Rotations

(a) Since $\delta x = \varepsilon \partial G/\partial p_x$, translation is generated by $G_x = p_x$. For the specified rotation, $G_z = L_z = xp_y - yp_x$, because

$$\{x, L_z\} = -y, \quad \{y, L_z\} = x, \quad \{p_x, L_z\} = -p_y, \quad \{p_y, L_z\} = p_x.$$

(b) Using $L_j = \epsilon_{jab} x_a p_b$,

$$\{x_i, L_j\} = \epsilon_{jai} x_a = \epsilon_{ija} x_a, \qquad \{p_i, L_j\} = -\epsilon_{jib} p_b = \epsilon_{ijb} p_b,$$

or, equivalently, $\{\boldsymbol{x}, L_j\} = \boldsymbol{e}_j \times \boldsymbol{x}$ and the same for $\boldsymbol{p}$. The last calculation gives

$$\{L_i, L_j\} = \epsilon_{ijk} L_k.$$

For $j = z$, these reproduce the signs in part (a).

(c) Since the kinetic term is rotationally invariant,

$$\dot{L}_z = \{L_z, H\} = \{L_z, V\} = yV_x - xV_y.$$

This vanishes precisely when the directional derivative of V along $(-y, x, 0)$ vanishes, i.e. when V is constant on the connected rotation orbits. For $V = \alpha(x^2 + y^2) + \beta z^2$, $y(2\alpha x) - x(2\alpha y) = 0$.

14. $SO(3)$, Its Lie Algebra, and Finite Rotations

(a)

$$[X_A, X_B]f = \{\{f, B\}, A\} - \{\{f, A\}, B\} = -\{f, \{A, B\}\} = -X_{\{A,B\}}f,$$

where Jacobi was used. Since $\{L_i, L_j\} = \epsilon_{ijk}L_k$, the generators close as $\mathfrak{so}(3)$. The anti-homomorphism sign comes from defining a right action on observables, $X_A f = \{f, A\}$; using $f \mapsto \{A, f\}$ reverses it.

(b) Split $\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$, with $\boldsymbol{x}_\parallel = \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{x}_0)$. The parallel part is constant, while $d^2\boldsymbol{x}_\perp/d\alpha^2 = \boldsymbol{n} \times (\boldsymbol{n} \times \boldsymbol{x}_\perp) = -\boldsymbol{x}_\perp$. The initial derivative is $\boldsymbol{n} \times \boldsymbol{x}_0$, giving Rodrigues' formula.

(c) The Baker–Campbell–Hausdorff expansion gives

$$e^{\varepsilon X_x} e^{\varepsilon X_y} e^{-\varepsilon X_x} e^{-\varepsilon X_y} = \exp\left(\varepsilon^2 [X_x, X_y] + O(\varepsilon^3)\right) = \exp\left(-\varepsilon^2 X_{L_z} + O(\varepsilon^3)\right).$$

Thus, for the order displayed, the net operation is a rotation about $-z$ through $\varepsilon^2 + O(\varepsilon^3)$. Reversing the commutator order reverses the sign.

15. Hidden Symmetry of the Kepler Problem

(a) Rotational invariance immediately gives $\{L_i, H\} = 0$. For a direct vector check,

$$\dot{\boldsymbol{p}} = -\frac{k\boldsymbol{r}}{r^3}, \quad \dot{\boldsymbol{L}} = 0, \quad \frac{d}{dt}\frac{\boldsymbol{r}}{r} = \frac{1}{\mu r}\left(\boldsymbol{p} - \frac{\boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{p})}{r^2}\right).$$

Using $\boldsymbol{r} \times (\boldsymbol{r} \times \boldsymbol{p}) = \boldsymbol{r}(\boldsymbol{r} \cdot \boldsymbol{p}) - r^2\boldsymbol{p}$,

$$\dot{\boldsymbol{A}} = \dot{\boldsymbol{p}} \times \boldsymbol{L} - \mu k \frac{d}{dt}(\boldsymbol{r}/r) = 0.$$

(b) Both terms in $\boldsymbol{A}$ are perpendicular to $\boldsymbol{L}$, so $\boldsymbol{A} \cdot \boldsymbol{L} = 0$. Also $\boldsymbol{p} \times \boldsymbol{L} = \boldsymbol{r}\, p^2 - \boldsymbol{p}(\boldsymbol{r} \cdot \boldsymbol{p})$, whence

$$(\boldsymbol{p} \times \boldsymbol{L})^2 = p^2 L^2, \qquad \frac{\boldsymbol{r}}{r} \cdot (\boldsymbol{p} \times \boldsymbol{L}) = \frac{L^2}{r}.$$

Therefore

$$A^2 = p^2 L^2 - \frac{2\mu k L^2}{r} + \mu^2 k^2 = 2\mu H L^2 + \mu^2 k^2.$$

Furthermore $\boldsymbol{A} \cdot \boldsymbol{r} = L^2 - \mu k r$. If ψ is the angle from $\boldsymbol{A}$,

$$r = \frac{L^2/(\mu k)}{1 + [A/(\mu k)]\cos\psi}, \qquad e = A/(\mu k).$$

(c) Expanding with canonical brackets and the Levi–Civita identity yields

$$\{L_i, A_j\} = \epsilon_{ijk} A_k, \qquad \{A_i, A_j\} = -2\mu H \epsilon_{ijk} L_k.$$

(A short route to the first is that $\boldsymbol{A}$ is built as a vector from $\boldsymbol{r}, \boldsymbol{p}$; the second then has the only allowed antisymmetric vector form, with its coefficient fixed by one component.) Since $\{H, A_i\} = \{H, L_i\} = 0$, for $H < 0$,

$$\{D_i, D_j\} = \epsilon_{ijk} L_k, \qquad \{L_i, D_j\} = \epsilon_{ijk} D_k.$$

Consequently

$$\{J_{\pm i}, J_{\pm j}\} = \epsilon_{ijk} J_{\pm k}, \qquad \{J_{+i}, J_{-j}\} = 0,$$

so the bound Kepler problem has $\mathfrak{so}(3) \oplus \mathfrak{so}(3) \cong \mathfrak{so}(4)$ dynamical symmetry.

16. Discrete Symmetry and Normal Modes

(a)

$$m\ddot{q}_j = -k(2q_j - q_{j-1} - q_{j+1}).$$

The cyclic relabeling leaves both sums in L unchanged, and three applications return every label: $C^3 = I$.

(b) Substitution gives

$$\omega_s^2 = \frac{k}{m}\left(2 - e^{2\pi i s/3} - e^{-2\pi i s/3}\right) = \frac{4k}{m}\sin^2\frac{\pi s}{3}.$$

Thus $\omega_0 = 0$ and $\omega_1 = \omega_2 = \sqrt{3k/m}$. The $s = 1, 2$ modes transform by the conjugate characters $e^{\pm 2\pi i/3}$. A real system pairs these conjugate one-dimensional complex representations into a two-dimensional real representation, forcing the degeneracy.

(c) Use the orthogonal transformation

$$Q_0 = \frac{q_0 + q_1 + q_2}{\sqrt{3}}, \quad Q_c = \frac{2q_0 - q_1 - q_2}{\sqrt{6}}, \quad Q_s = \frac{q_1 - q_2}{\sqrt{2}},$$

and the identical formulas with $q \to p$, $Q \to P$. Orthogonality gives $\{Q_a, P_b\} = \delta_{ab}$ and all Q–Q and P–P brackets vanish. Hence

$$H = \frac{P_0^2 + P_c^2 + P_s^2}{2m} + \frac{3k}{2}(Q_c^2 + Q_s^2).$$

Q_0 is the free center-of-mass coordinate; (Q_c, Q_s) are the degenerate pair.

17. Canonical Transformations from a Generator

(a) Since $\{Q, G\} = Q$ and $\{P, G\} = -P$,

$$Q = e^{\varepsilon} q, \qquad P = e^{-\varepsilon} p.$$

Therefore $\{Q, P\} = e^{\varepsilon}e^{-\varepsilon} = 1$ and $dQ \wedge dP = dq \wedge dp$.

(b) Evolving any observable gives $\dot{f}_\varepsilon = \{f_\varepsilon, G\}$. The Leibniz/derivation property plus Jacobi gives

$$\frac{d}{d\varepsilon}\{f_\varepsilon, g_\varepsilon\} = \{\{f_\varepsilon, G\}, g_\varepsilon\} + \{f_\varepsilon, \{g_\varepsilon, G\}\} = \{\{f_\varepsilon, g_\varepsilon\}, G\}.$$

This is the same evolution equation and initial value as $(\{f, g\})_\varepsilon$, proving equality.

(c) Let $A = (4E/k)^{1/4}$ be the turning amplitude and choose $e^{\varepsilon} = A^{-1}$, so the canonical variables are $Q = q/A$ and $P = Ap$; the orbit has unit turning point in Q. Energy conservation also supplies the physical momentum scale $p_E = \sqrt{2mE}$. Hamilton's equations become dimensionless after writing

$$q = AQ, \qquad p = p_E\Pi, \qquad t = t_E\tau, \qquad t_E = \frac{mA}{p_E}.$$

The dimensionless orbit has an energy-independent period, so

$$T(E) \propto t_E = \frac{m(4E/k)^{1/4}}{(2mE)^{1/2}} = m^{1/2}k^{-1/4}E^{-1/4}.$$

Only an energy-independent numerical factor (a beta-function integral) is omitted.

18. Rigid-Body Dynamics from the Angular-Momentum Algebra

(a) The body-frame bracket gives Euler's equations

$$\dot{L}_1 = \left(\frac{1}{I_3} - \frac{1}{I_2}\right)L_2L_3, \quad \dot{L}_2 = \left(\frac{1}{I_1} - \frac{1}{I_3}\right)L_3L_1, \quad \dot{L}_3 = \left(\frac{1}{I_2} - \frac{1}{I_1}\right)L_1L_2.$$

Multiplying by L_i and summing makes the three cubic terms cancel, so $dL^2/dt = 0$. Likewise $\dot{H} = \{H, H\} = 0$.

(b) About axis 1, set $L_1 \simeq L$ and keep (L_2, L_3) small. The product of the two linear coefficients is negative, hence stable, with

$$\omega_1^2 = L^2 \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\left(\frac{1}{I_1} - \frac{1}{I_2}\right).$$

About axis 3 the motion is also stable, with

$$\omega_3^2 = L^2 \left(\frac{1}{I_1} - \frac{1}{I_3}\right)\left(\frac{1}{I_2} - \frac{1}{I_3}\right).$$

About axis 2 the coefficient product is positive, so perturbations grow exponentially: the intermediate-axis rotation is unstable.

(c) If $I_1 = I_2 = I$, then $\dot{L}_3 = 0$. Define

$$\Omega = \left(\frac{1}{I} - \frac{1}{I_3}\right) L_3.$$

The other equations are $\dot{L}_1 = -\Omega L_2$, $\dot{L}_2 = \Omega L_1$, hence

$$L_1 = A\cos(\Omega t + \phi), \qquad L_2 = A\sin(\Omega t + \phi), \qquad L_3 = \text{constant}.$$

This is the standard one-parameter group $SO(2)$ acting on the (L_1, L_2) plane.

Design references

The sequence and emphasis were checked against the official The Theoretical Minimum classical mechanics course, especially Lectures 1–8, and against the proof-and-calculation style of MIT OCW 8.223 Classical Mechanics II and MIT OCW 8.09 Classical Mechanics III problem sets. The problems here are newly composed and solutions are self-contained.