Problem 1: Newtonian Motion as a Flow

A particle of mass m moves on a line in the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 + \frac{\lambda}{4}x^4, \qquad \lambda > 0.$$

(a) Starting from Newton's law, introduce $v = \dot{x}$ and write an autonomous first-order system $\dot{\boldsymbol{z}} = \boldsymbol{F}(\boldsymbol{z})$ on state space $\boldsymbol{z} = (x, v)$. Find all fixed points and linearize the flow at each one.

(b) Prove directly from the first-order equations that

$$E(x, v) = \frac{1}{2}mv^2 + V(x)$$

is constant. Hence derive an integral expression for the period $T(A)$ of an orbit with amplitude $A > 0$.

(c) For $\lambda A^2 \ll m\omega^2$, expand the integral through first order in λ and show that

$$T(A) = \frac{2\pi}{\omega}\left(1 - \frac{3\lambda A^2}{8m\omega^2} + O(\lambda^2 A^4)\right).$$

Use the remaining space for derivations.

Problem 2: From Stationary Action to a Boundary-Value Solution

Let $S[q] = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$ and consider $q_\varepsilon = q + \varepsilon\eta$ with $\eta(t_1) = \eta(t_2) = 0$.

(a) Compute $dS[q_\varepsilon]/d\varepsilon|_0$, integrate by parts, and derive the Euler–Lagrange equation. State precisely where the endpoint condition is used.

(b) Apply the result to a particle in a uniform field,

$$L = \frac{1}{2}m\dot{y}^2 - mgy, \qquad y(0) = 0, \quad y(T) = h.$$

Find the stationary path $y_\star(t)$.

(c) Write $y = y_\star + \eta$ with $\eta(0) = \eta(T) = 0$ and prove the exact identity

$$S[y] - S[y_\star] = \frac{m}{2}\int_0^T \dot{\eta}^2\, dt.$$

Does "stationary" mean minimum for this example? Your answer must follow from the identity, not from a verbal appeal to least action.

Use the remaining space for derivations.

Problem 3: Generalized Coordinates: Bead on a Rotating Hoop

A bead of mass m slides without friction on a vertical circular hoop of radius R. The hoop rotates at constant angular speed Ω about its vertical diameter. Let θ be measured from the downward vertical. Thus

$$\boldsymbol{r} = R\big(\sin\theta\cos\Omega t,\ \sin\theta\sin\Omega t,\ -\cos\theta\big).$$

(a) Compute $\dot{\boldsymbol{r}}^{,2}$ and construct $L(\theta,\dot\theta)$, eliminating the Cartesian constraint automatically.

(b) Derive the equation of motion and write it as $mR^2\ddot\theta = -dV_{\text{eff}}/d\theta$. Determine all equilibria and the parameter value at which new equilibria appear.

(c) Linearize about every equilibrium and classify its stability. For each stable equilibrium, find the small-oscillation frequency.

Use the remaining space for derivations.

Problem 4: Constraints and Multipliers: A Sliding Wedge

A block of mass m slides without friction down a wedge of mass M and angle α. The wedge can move horizontally on a frictionless floor. Use laboratory coordinates (x, y) for the block and X for the wedge. The constraint is

$$f(x, y, X) = y + (x - X)\tan\alpha - y_0 = 0.$$

(a) Starting from $L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$ and the multiplier equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \Lambda\frac{\partial f}{\partial q_i},$$

derive equations for x, y, X, and Λ.

(b) Eliminate the constraint and solve for the horizontal wedge acceleration $\ddot{X}$ and the bead's acceleration $\ddot{s}$ down the incline, where $x = X + s\cos\alpha$ and $y = y_0 - s\sin\alpha$.

(c) Determine the normal force N from Λ, and verify explicitly that total horizontal momentum and total mechanical energy are conserved.

Use the remaining space for derivations.

Problem 5: Noether's Theorem in Calculational Form

Suppose an infinitesimal transformation $q_i \mapsto q_i + \varepsilon X_i(q,t)$ changes the Lagrangian by a total derivative,

$$\delta L = \varepsilon \frac{d}{dt} F(q,t).$$

(a) Using the Euler–Lagrange equations, prove that

$$Q = \sum_i p_i X_i - F, \qquad p_i = \frac{\partial L}{\partial \dot{q}_i},$$

is conserved on every classical trajectory.

(b) For N particles with

$$L = \sum_{a=1}^{N} \frac{m_a}{2} \dot{\boldsymbol{r}}_a^{\,2} - \sum_{a<b} U_{ab}(|\boldsymbol{r}_a - \boldsymbol{r}_b|),$$

apply the formula to a common translation $\delta \boldsymbol{r}_a = \varepsilon \boldsymbol{a}$ and a common infinitesimal rotation $\delta \boldsymbol{r}_a = \varepsilon \boldsymbol{n} \times \boldsymbol{r}_a$. Derive the corresponding vector charges.

(c) Apply a Galilean boost $\delta \boldsymbol{r}_a = \varepsilon \boldsymbol{u}, t$. Show that $\delta L = dF/dt$, determine F, and obtain the conserved boost charge.

Use the remaining space for derivations.

Problem 6: Central Forces: Reduction Before Solution

A particle of reduced mass μ moves in $V(r) = -k/r$.

(a) Starting with the planar polar Lagrangian, identify the cyclic coordinate, eliminate $\dot{\phi}$ in favor of its conserved momentum ℓ, and derive

$$\frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) = E, \qquad V_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} - \frac{k}{r}.$$

(b) Find the circular-orbit radius and energy. Compute the frequency of small radial oscillations and compare it with the angular frequency of the circular orbit.

(c) With $u(\phi) = 1/r$, derive Binet's equation and solve it to obtain

$$r(\phi) = \frac{p}{1 + e\cos(\phi - \phi_0)}.$$

Express p and e in terms of (ℓ, E, μ, k) and classify the orbit by E.

Use the remaining space for derivations.

Problem 7: One System in Three Languages: The Double Pendulum

Two equal point masses m are joined by massless rods of equal length ℓ in a vertical plane. Angles θ_1, θ_2 are measured from the downward vertical.

(a) Derive the exact Lagrangian

$$L = \frac{m\ell^2}{2}\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) + mg\ell(2\cos\theta_1 + \cos\theta_2).$$

(b) Linearize about the downward equilibrium. Write the equations as $\mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = 0$ and find both normal-mode frequencies and eigenvectors.

(c) For the linearized theory, compute $\boldsymbol{p} = \mathbf{M}\dot{\boldsymbol{\theta}}$, perform the Legendre transform, and write Hamilton's equations. Verify that eliminating $\boldsymbol{p}$ returns the matrix equation in part (b).

Use the remaining space for derivations.

Problem 8: When Canonical Momentum Is Not $m\dot{q}$

Consider

$$L(q,\dot{q}) = \frac{\alpha}{4}\dot{q}^4 - \frac{k}{2}q^2, \qquad \alpha, k > 0.$$

(a) Derive the Euler–Lagrange equation and the conserved energy $E = \dot{q}\,\partial L/\partial\dot{q} - L$.

(b) Compute the canonical momentum, invert for $\dot{q}(p)$ using the real cube root, and perform the Legendre transform to obtain $H(q,p)$. Check both Hamilton equations against part (a).

(c) For an orbit of amplitude A, reduce the period to a beta-function integral and show

$$T(A) = 4\left(\frac{3\alpha}{2k}\right)^{1/4} A^{1/2} \int_0^1 \frac{du}{(1-u^2)^{1/4}}.$$

Evaluate the integral in terms of Gamma functions and determine how the frequency scales with A.

Use the remaining space for derivations.

Problem 9: Hamiltonian Flow of the Harmonic Oscillator

For $H = p^2/(2m) + m\omega^2 q^2/2$:

(a) Derive and solve Hamilton's equations for arbitrary (q_0, p_0). Write the result as

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = S(t) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix}.$$

(b) With $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, prove $S(t)^T J S(t) = J$ and $\det S(t) = 1$. Verify the group law $S(t+s) = S(t)S(s)$.

(c) Rescale $Q = \sqrt{m\omega}\, q$, $P = p/\sqrt{m\omega}$. Show that the flow becomes a rotation. Determine the phase-space area enclosed by the orbit $H = E$ and compute the action variable $I = (2\pi)^{-1} \oint p\, dq$.

Use the remaining space for derivations.

Problem 10: Pendulum Phase Portrait and Separatrix

For a simple pendulum,

$$H(\theta, p) = \frac{p^2}{2m\ell^2} + mg\ell(1 - \cos\theta).$$

(a) Find all fixed points on the cylinder $(\theta \bmod 2\pi, p)$ and classify them by linearization.

(b) The separatrix has energy $E_s = 2mg\ell$. Solve $p(\theta)$ on it and integrate Hamilton's equations to show that one branch can be written

$$\theta(t) = 4\arctan e^{\omega_0(t-t_0)} - \pi, \qquad \omega_0 = \sqrt{g/\ell}.$$

(c) For a libration of turning angle θ_0, reduce the period to

$$T = 4\sqrt{\frac{\ell}{g}}\,K\!\left(\sin\frac{\theta_0}{2}\right), \qquad K(k) = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2\sin^2\psi}}.$$

Explain the divergence as $\theta_0 \to \pi$ by extracting the local behavior of the integral.

Use the remaining space for derivations.

Problem 11: Liouville's Theorem and Its Failure for Damping

For canonical coordinates $\boldsymbol{z} = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, let $\dot{\boldsymbol{z}} = \boldsymbol{v}_H(\boldsymbol{z}, t)$ be Hamiltonian flow.

(a) Compute $\nabla_{\boldsymbol{z}} \cdot \boldsymbol{v}_H$ and prove it vanishes for every C^2 Hamiltonian, including explicitly time-dependent ones.

(b) Let $M(t) = \partial \boldsymbol{z}(t)/\partial \boldsymbol{z}(0)$ be the tangent map. Using Jacobi's formula

$$\frac{d}{dt} \det M = (\det M)\, \mathrm{tr}(M^{-1}\dot{M}),$$

prove $\det M(t) = 1$ and hence phase volume is preserved.

(c) Compare the damped oscillator $\dot{q} = p/m$, $\dot{p} = -m\omega^2 q - \gamma p$. Compute its divergence and the exact evolution of an infinitesimal phase-space area. Show why no time-independent canonical Hamiltonian $H(q, p)$ can produce these equations.

Use the remaining space for derivations.

Problem 12: Poisson Brackets from Hamilton's Equations

Define

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right).$$

(a) Derive $df/dt = \partial f/\partial t + \{f, H\}$ from Hamilton's equations. Establish bilinearity, antisymmetry, and the Leibniz rule directly from the definition.

(b) In matrix notation with $\boldsymbol{z} = (\boldsymbol{q}, \boldsymbol{p})$ and $J = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, show $\{f, g\} = (\nabla f)^T J \nabla g$. Use equality of mixed partials to prove the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

(c) If A and B have no explicit time dependence and each is conserved, prove that $\{A, B\}$ is also conserved. Apply this to check closure of the conserved quantities for an isotropic oscillator.

Use the remaining space for derivations.

Problem 13: Generators of Translations and Rotations

An infinitesimal canonical transformation generated by G is defined by $\delta f = \varepsilon \{f, G\}$.

(a) Find generators G_x and G_z that produce, respectively, a translation $x \mapsto x + \varepsilon$ and a rotation about the z axis,

$$\delta x = -\varepsilon y, \qquad \delta y = \varepsilon x, \qquad \delta z = 0,$$

including the induced transformations of momenta.

(b) For $L_i = \epsilon_{ijk} x_j p_k$, calculate

$$\{x_i, L_j\}, \qquad \{p_i, L_j\}, \qquad \{L_i, L_j\}.$$

Fix all signs by checking the z rotation in part (a).

(c) For $H = \boldsymbol{p}^2/(2m) + V(\boldsymbol{x})$, prove that L_z is conserved iff V is invariant under rotations about z (locally, on a connected domain). Verify the criterion for $V = \alpha(x^2 + y^2) + \beta z^2$.

Use the remaining space for derivations.

Problem 14: $SO(3)$, Its Lie Algebra, and Finite Rotations

Let $A(\boldsymbol{a}) = a_i L_i$ and write $X_A f = \{f, A\}$.

(a) Use the Jacobi identity to prove

$$[X_A, X_B]f = -X_{\{A,B\}}f.$$

Then use $\{L_i, L_j\} = \epsilon_{ijk} L_k$ to identify the Lie algebra represented by the rotation generators (explain the displayed minus sign by the convention $X_A f = \{f, A\}$).

(b) For a unit vector $\boldsymbol{n}$, solve

$$\frac{d\boldsymbol{x}}{d\alpha} = \{\boldsymbol{x}, \boldsymbol{n} \cdot \boldsymbol{L}\} = \boldsymbol{n} \times \boldsymbol{x}$$

to derive Rodrigues' formula

$$\boldsymbol{x}(\alpha) = \boldsymbol{x}_0 \cos\alpha + (\boldsymbol{n} \times \boldsymbol{x}_0) \sin\alpha + \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{x}_0)(1 - \cos\alpha).$$

(c) Let $R_i(\varepsilon) = e^{\varepsilon X_{L_i}}$. Expand the group commutator

$$R_x(\varepsilon) R_y(\varepsilon) R_x(-\varepsilon) R_y(-\varepsilon)$$

through $O(\varepsilon^2)$ and identify the resulting infinitesimal rotation.

Use the remaining space for derivations.

Problem 15: Hidden Symmetry of the Kepler Problem

For

$$H = \frac{\boldsymbol{p}^2}{2\mu} - \frac{k}{r}, \qquad \boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}, \qquad \boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{L} - \mu k \, \frac{\boldsymbol{r}}{r},$$

use canonical Poisson brackets in three dimensions.

(a) Compute $\{L_i, H\}$ and $\{A_i, H\}$, proving conservation of both vectors.

(b) Prove

$$\boldsymbol{A} \cdot \boldsymbol{L} = 0, \qquad A^2 = \mu^2 k^2 + 2\mu H L^2,$$

and derive the conic orbit equation by taking $\boldsymbol{A} \cdot \boldsymbol{r}$.

(c) Establish

$$\{L_i, A_j\} = \epsilon_{ijk} A_k, \qquad \{A_i, A_j\} = -2\mu H \, \epsilon_{ijk} L_k.$$

For $H < 0$, rescale $\boldsymbol{D} = \boldsymbol{A}/\sqrt{-2\mu H}$ and show that $\boldsymbol{J}_\pm = (\boldsymbol{L} \pm \boldsymbol{D})/2$ generate two commuting $\mathfrak{so}(3)$ algebras.

Use the remaining space for derivations.

Problem 16: Discrete Symmetry and Normal Modes

Three identical masses m lie on a ring and are joined by identical springs of constant k. Let q_j be small displacements and impose $q_{j+3} = q_j$:

$$L = \frac{m}{2}\sum_{j=0}^{2}\dot{q}_j^2 - \frac{k}{2}\sum_{j=0}^{2}(q_{j+1} - q_j)^2.$$

(a) Derive the equations of motion. Show that the cyclic permutation $C(q_0, q_1, q_2) = (q_1, q_2, q_0)$ is a symmetry and satisfies $C^3 = I$.

(b) Seek complex modes $q_j(t) = Ae^{i(2\pi sj/3 - \omega_s t)}$, $s = 0, 1, 2$. Find all frequencies and explain the degeneracy using the irreducible characters $1, e^{2\pi i/3}, e^{4\pi i/3}$ of C_3.

(c) Construct a real canonical change of variables $(\boldsymbol{q}, \boldsymbol{p}) \mapsto (Q_0, Q_c, Q_s, P_0, P_c, P_s)$ that diagonalizes the Hamiltonian. Verify its canonicity from Poisson brackets and write the diagonal H.

Use the remaining space for derivations.

Problem 17: Canonical Transformations from a Generator

Let $G(q,p) = qp$ and define a one-parameter transformation by

$$\frac{dQ}{d\varepsilon} = \{Q, G\}, \qquad \frac{dP}{d\varepsilon} = \{P, G\}, \qquad (Q,P)|_{\varepsilon=0} = (q,p).$$

(a) Solve for (Q,P) and verify directly that $\{Q,P\} = 1$ and $dQ \wedge dP = dq \wedge dp$.

(b) Prove generally that the flow generated by any smooth G preserves Poisson brackets:

$$\frac{d}{d\varepsilon}\left(\{f_\varepsilon, g_\varepsilon\} - \big(\{f,g\}\big)_\varepsilon\right) = 0,$$

using the Jacobi identity.

(c) Apply the scaling to $H = p^2/(2m) + kq^4/4$. Choose ε so that an energy-E orbit has unit turning point in Q, and deduce by rescaling Hamilton's equations that $T(E) \propto E^{-1/4}$. Determine the remaining dependence on m and k.

Use the remaining space for derivations.

Problem 18: Rigid-Body Dynamics from the Angular-Momentum Algebra

In principal body axes, a free rigid body has

$$H = \frac{L_1^2}{2I_1} + \frac{L_2^2}{2I_2} + \frac{L_3^2}{2I_3}, \qquad \{L_i, L_j\} = -\epsilon_{ijk} L_k$$

(the minus sign is the body-frame convention). Assume $I_1 < I_2 < I_3$.

(a) Use $\dot{L}_i = \{L_i, H\}$ to derive Euler's equations. Prove directly that L^2 and H are conserved.

(b) Linearize about steady rotation around each principal axis. Determine which axes are stable and find the small-perturbation frequency for each stable rotation of magnitude L.

(c) For the symmetric top $I_1 = I_2 = I \neq I_3$, solve Euler's equations explicitly for $L_1(t), L_2(t), L_3(t)$ in the body frame. Identify the one-parameter rotation group acting on (L_1, L_2).

End of problem set. Use the remaining space for derivations.